

Phase-ordering and persistence: relative effects of space-discretization, chaos, and anisotropy

Julien Kockelkoren, Anaël Lemaître, and Hugues Chaté

*CEA, Service de Physique de l'Etat Condensé
Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France*

Abstract

The peculiar phase-ordering properties of a lattice of coupled chaotic maps studied recently (A. Lemaître & H. Chaté, *Phys. Rev. Lett.* **82**, 1140 (1999)) are revisited with the help of detailed investigations of interface motion. It is shown that “normal”, curvature-driven-like domain growth is recovered at larger scales than considered before, and that the persistence exponent seems to be universal. Using generalized persistence spectra, the properties of interface motion in this deterministic, chaotic, lattice system are found to “interpolate” between those of the two canonical reference systems, the (probabilistic) Ising model, and the (deterministic), space-continuous, time-dependent Ginzburg-Landau equation.

1 Introduction

Following a quench at low temperature, bistable “ferromagnetic” systems usually exhibit domain coarsening dynamics. This phase separation process has been observed in various experimental setups, as well as in numerous model systems [1]. The Ising model and its continuous counterpart, the so-called time-dependent Ginzburg-Landau (TDGL) equation are usually taken as paradigms of non-conservative coarsening dynamics [2]. But many other systems, e.g. the diffusion equation, also exhibit phase-ordering dynamics once suitable “phases” are defined (for the diffusion equation, one can for instance consider the sign of a zero-mean field). Spatially-extended chaotic systems such as coupled map lattices (CMLs) can also exhibit coarsening transients, after which one chaotic phase dominates another, leaving the system in a long-range ordered state that is usually accompanied by a non-trivial evolution of spatially-averaged quantities [3,4].

The standard theory of domain coarsening predicts that, in general, the correlation length $L(t)$ grows algebraically in time, $L(t) \sim t^{1/z}$ [1,2]. Moreover, the exponent usually takes the value $z = 2$ in systems with a non-conserved, scalar order parameter when the domain growth is driven by curvature. This universality of domain growth processes is now well established. Another important quantifier of phase ordering dynamics is persistence [5], defined, e.g., as the fraction of space that has remained in the same phase since some given initial time t_0 . Persistence is seen to decay algebraically $p(t) \propto t^{-\theta}$ with exponent θ in systems with algebraic domain growth, reflecting the stationarity of two-time correlations expressed in logarithmic time. As a matter of fact, the correlation length $L(t)$ provides a better, “natural” measure of time, by which persistence decays as $p(t) \propto L(t)^{-\bar{\theta}}$ (with $\bar{\theta} = z\theta$) even in systems with, say, logarithmic growth of domains. The (degree of) universality of persistence exponents is still largely an open question today. Even models as close to each other as the zero-temperature Ising model and the TDGL equation—they share the same exponent $z = 2$, the same structure function, the same Fisher-Huse exponent—seem to possess different persistence exponents (currently available measurements give $\theta = 0.20$ for TDGL [6,7] and 0.22 for Ising [8,7]).

In fact, since it depends on the whole history of each point in the space, persistence is a rather complex, non-local quantity. Exact or approximate values of persistence exponents are usually not simple numbers. Understanding the extent to which persistence properties are universal is one of the challenging problems of modern statistical physics. It is also important from an experimental point of view, since persistence is an easily measurable quantifier of phase ordering. One can hope to bring new light to this problem by studying non-conventional models like CMLs. In a sense, they are not constrained as are traditional models: their local dynamics cannot be rigorously reduced to that of standard models: there is no Hamiltonian, no detailed balance, and they have been shown (numerically) to exhibit Ising-like phase transitions with critical exponents that are significantly different from those of the Ising model [9].

In [4], domain growth has been studied in simple CMLs. Numerical simulations showed the expected scaling behavior, but with exponents z and θ continuously varying with the strength of the diffusive coupling between chaotic units, although $\bar{\theta}$ was found to be universal. It was also found that “normal”, TDGL, values were recovered in the continuous space limit of CMLs, suggesting that a role is played by lattice effects.

In this article, we come back to these somewhat surprising results and study in some detail the dynamics of the interfaces delimiting domains, in order to unravel the origin of the peculiar behavior observed on more global quantities, such as $L(t)$.

2 Domain growth and interface dynamics in chaotic CMLs

We consider $d = 2$ dimensional square lattices \mathcal{L} of diffusively-coupled identical maps S_μ acting on real variables $(X_{\vec{r}})_{\vec{r} \in \mathcal{L}}$:

$$X_{\vec{r}}^{t+1} = (1 - 2dg)S_\mu(X_{\vec{r}}^t) + g \sum_{\vec{e} \in \mathcal{V}} S_\mu(X_{\vec{r}+\vec{e}}^t) , \quad (1)$$

where \mathcal{V} is the set of $2d$ nearest neighbors \vec{e} of site $\vec{0}$. For simplicity, we present results obtained for the piecewise linear, odd, local map S_μ defined by:

$$S_\mu(X) = \begin{cases} \mu X & \text{if } X \in [-1/3, 1/3] \\ 2\mu/3 - \mu X & \text{if } X \in [1/3, 1] \\ -2\mu/3 - \mu X & \text{if } X \in [-1, -1/3] . \end{cases} \quad (2)$$

Choosing $\mu \in [1, 2]$ guarantees that the local map has two symmetric invariant intervals, allowing us to define “spins” as $\sigma_{\vec{r}} = \text{sign}(X_{\vec{r}})$. The deterministic system thus defined is similar to the Ising model at zero temperature in the sense that local variables can flip only when crossed by an interface.

Take $\mu = 1.9$ and consider different choices for coupling strength g . Two cases are illustrated in Fig. 1: at small g , the evolution leads to blocked clusters of the two “phases” corresponding to the two invariant intervals (which “shrink” but are preserved by the linear diffusive coupling). Domain walls are pinned between lattice sites; the system is multistable. On the contrary, for larger values of the coupling, domain coarsening never stops, leading to the emergence of long-range order.

Thus, there exists a threshold value g_e separating these regimes: domain growth is expected to slow down as g goes to g_e from above, corresponding to a gradual “freezing” of interface dynamics.

2.1 Previous results

We first review some of the results presented in [4] for the sake of completeness.

Starting from random initial conditions with exactly zero magnetization (for the “spin” variables $\sigma_{\vec{r}} = \text{sign}(X_{\vec{r}})$), we measured $L(t)$ defined by the two-point correlation function at mid-height, as well as $p(t)$, the persistent fraction of sites since $t_0 = 0$. Fig. 2 shows the results obtained from single runs on lattices of 2048^2 sites simulated up to time $t = 10^4$ (at later times, finite-size

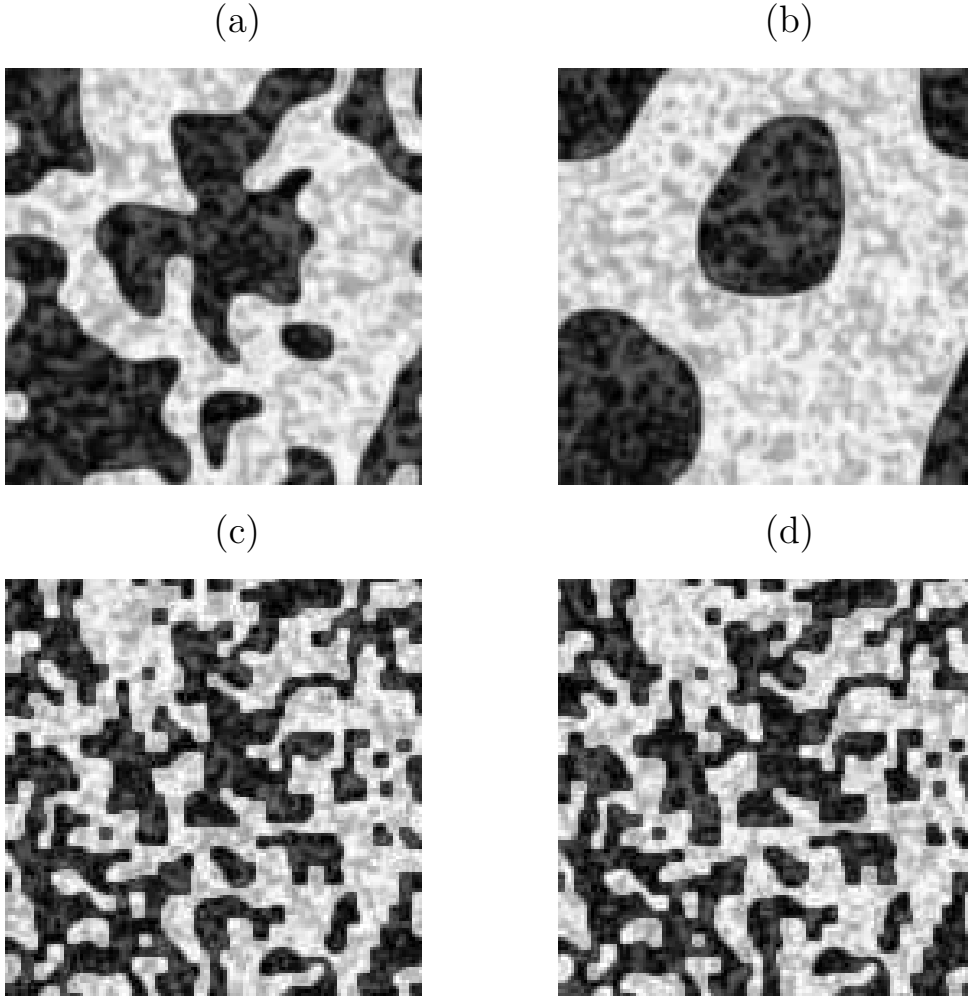


Fig. 1. Snapshots of the $d = 2$ CML with local map (2). Lattice of 128^2 sites, grey scale from $X = -1$ (white) to $X = 1$ (black), uncorrelated initial conditions. (a,b): transient leading to complete ordering at $g = 0.2 > g_e$, $t = 100$ and 1000 ; (c,d): blocked state at $g = 0.15 < g_e$, $t = 1000$ and 2000 .

fluctuations become too important). Algebraic growth of $L(t)$ and decay of $p(t)$ is observed, but with effective exponents z and θ which vary with g , the coupling strength. The decay of θ with g is best fitted with some power-law dependence which defines the threshold coupling $g_e \simeq 0.169(1)$.

Normal, $z = 2$ coarsening is only recovered when one approaches the continuous-space limit of CMLs which, in practice, consists in applying the coupling operator many times. In this limit, the persistence exponent θ approaches the value known for the TDGL equation, i.e., $\theta \simeq 0.20$. The underlying lattice thus seems to influence the propagation of fronts in the system, by discretization and/or anisotropy effects.

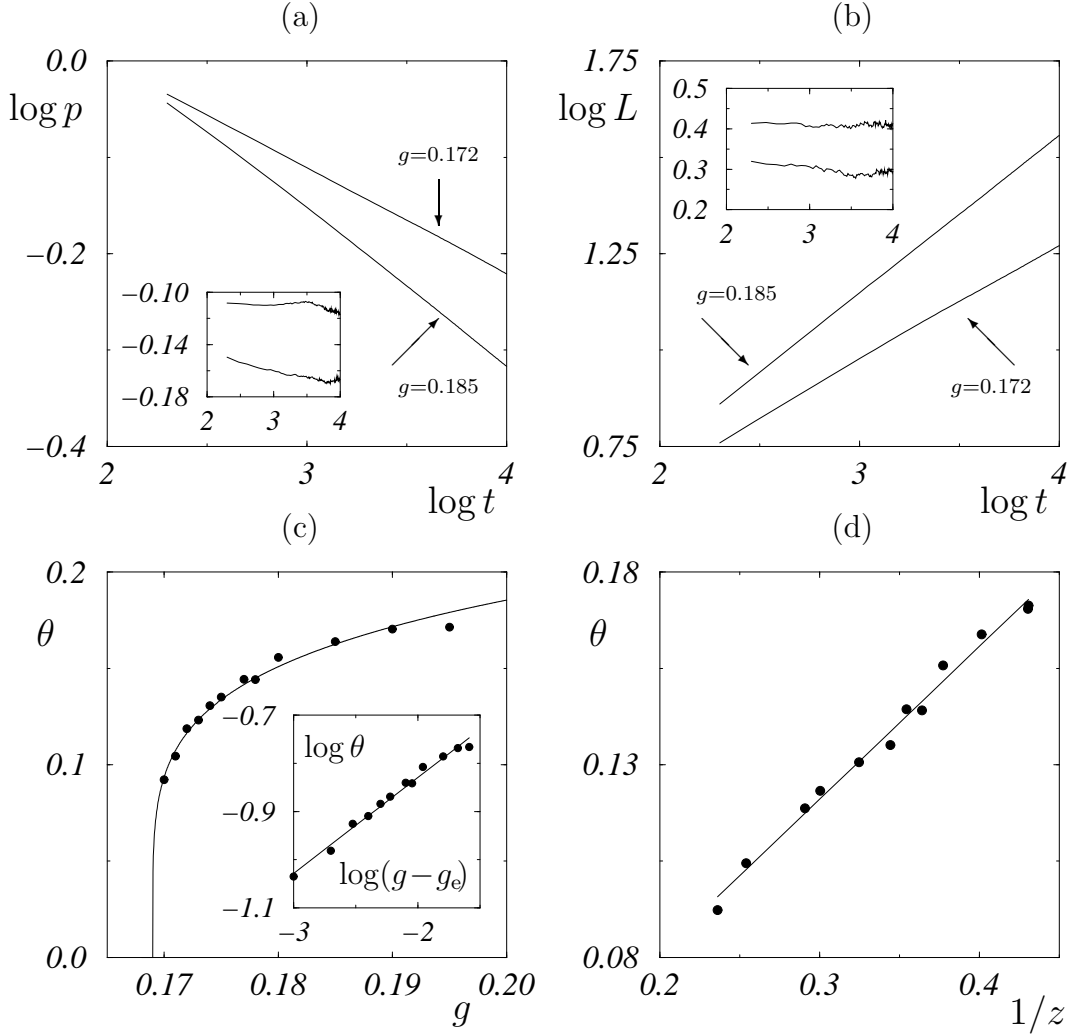


Fig. 2. Domain coarsening on a lattice of 2048^2 sites, up to time $t = 10^4$. (a): log-log plot of $p(t)$ vs time for two values of the coupling strength ($g = 0.172, 0.185$); local slope in insert (b): $L(t)$ in log-log for the same runs with local slope in insert (c): exponent θ as a function of the coupling strength g , measured from plots similar to (a). (d): linear dependence of persistence exponent θ with $1/z$.

2.2 Interface dynamics

Anisotropic curvature-driven domain growth can usually be defined by the following expression for the normal velocity of the interface [10]:

$$v_n = M(\varphi)[\sigma(\varphi) + \sigma''(\varphi))\kappa + h] \quad (3)$$

where $M(\varphi)$ is the mobility, $\sigma(\varphi)$ the surface tension, κ the local curvature, h an “external field”, and φ an angle specifying the orientation of the vector normal to the interface in space.

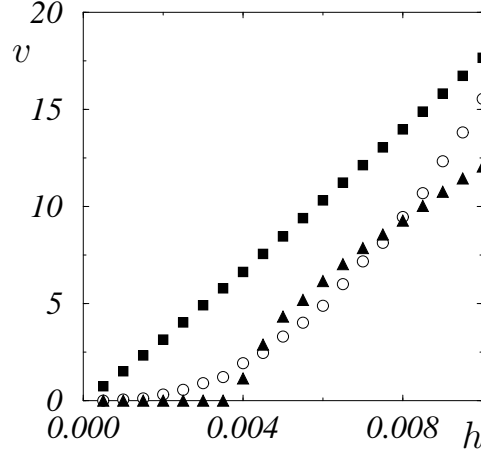


Fig. 3. Normal velocity v_n of inclined straight interfaces under the “external field” h for orientations $\varphi = 0$ (triangles), $\varphi = \frac{1}{4}\pi$ (circles) and $\varphi = \arctan(3/8)$ (squares) (for $g = 0.2$, lattices of linear size $\simeq 500$, from runs of 1000 iterations).

It is thus important to know whether these quantities can be defined in the CMLs of interest here, and, in the affirmative case, to try to estimate them in order to investigate possible discrepancies with Eq.(3).

The mobility is usually measured as the response of a flat interface ($\kappa = 0$) to the driving field h . Previous work has indicated that the mobility may not be defined in deterministic lattice systems [12,11]. This is also the case of the CMLs studied here. Replacing the evolution equation (1) by:

$$X_{\vec{r}}^{t+1} = h + (1 - h)((1 - 2dg)S_{\mu}(X_{\vec{r}}^t) + g \sum_{\vec{e} \in \mathcal{V}} S_{\mu}(X_{\vec{r}+\vec{e}}^t)) , \quad (4)$$

one observes moving interfaces for large-enough h , but the velocity may not be constant, and the h dependence of the average velocity is not always linear. Moreover, there generally exists a finite critical field value below which the interface does not move (Fig. 3). In fact, the mobility seems to be ill-defined for all rational values of $\tan \varphi$. Thus the mobility of the interfaces of the CML studied here is not well defined, a further indication of the special character of domain growth.

Another typical experiment to try to assess the validity of Eq. (3) is to study the fate of droplets of one phase immersed in the other phase. In anisotropic situations such as those described by Eq. (3), one expects, from some circular initial droplet, a transient stage during which the droplet takes its asymptotic shape and then shrinks self-similarly, its volume decreasing linearly with time; its lifetime is therefore proportional to its initial volume [13]. This is roughly what is observed with the CML studied here, except that the expected scaling is recorded only for droplets of large initial radius (Fig.4). For smaller size, one finds that the lifetime of droplets scales like $r^{2\alpha}$, with $1/2\alpha \simeq 1/z$, where

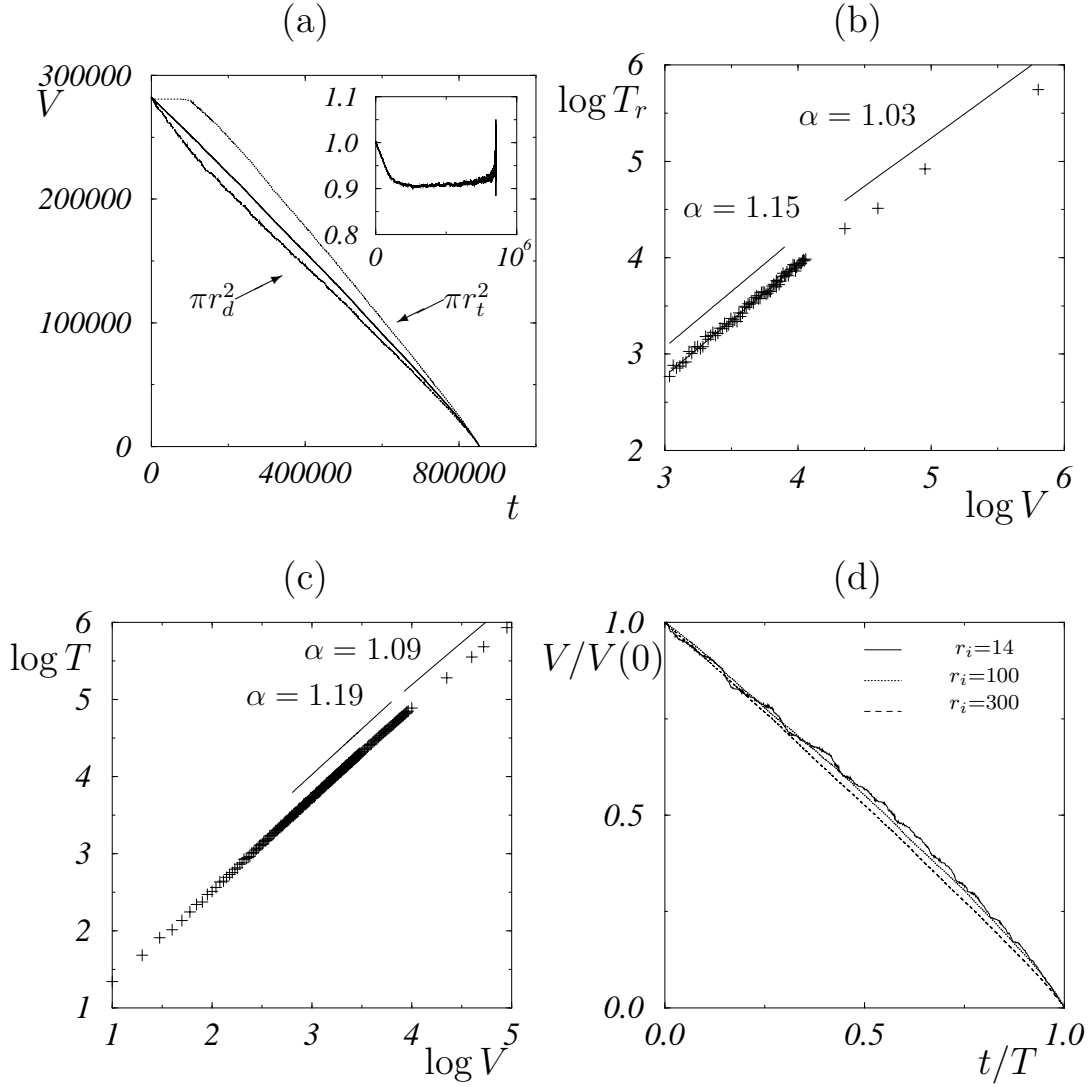


Fig. 4. Droplet evolution. (a) Time evolution of a circular droplet: volume V , πr_t^2 , and πr_d^2 vs time, where r_t and r_d are radii along, respectively, lattice and diagonal directions. Inset: ratio r_d/r_t of the two radii. Radius r_t remains initially constant during a transient “rearrangement” time. (b) Scaling of the rearrangement time T_r with initial volume of the droplet. The fit for “small” values of V (up to 10^4) yields an exponent $\alpha = 1.15$, while “normal” scaling is recovered for large values ($\alpha \simeq 1$) (c) Scaling of the lifetime T with initial volume. The fit for V yields $\alpha = 1.19$, while $\alpha = 1.09$, not far from 1, for large values of V . (d) For different initial volumes: rescaled volumes $V(t)/V(0)$ as a function of rescaled time t/T . The larger the initial radius, the straighter the curves.

z is the growth exponent measured in Fig.2. Interestingly, the crossover size is approximately of the order of the typical size of domains at the end of the runs shown in Fig.2.

This suggests that the non-trivial scalings laws reported in Fig.2 might only be transient, ultimately leaving place to normal (curvature-driven-like) phase ordering.

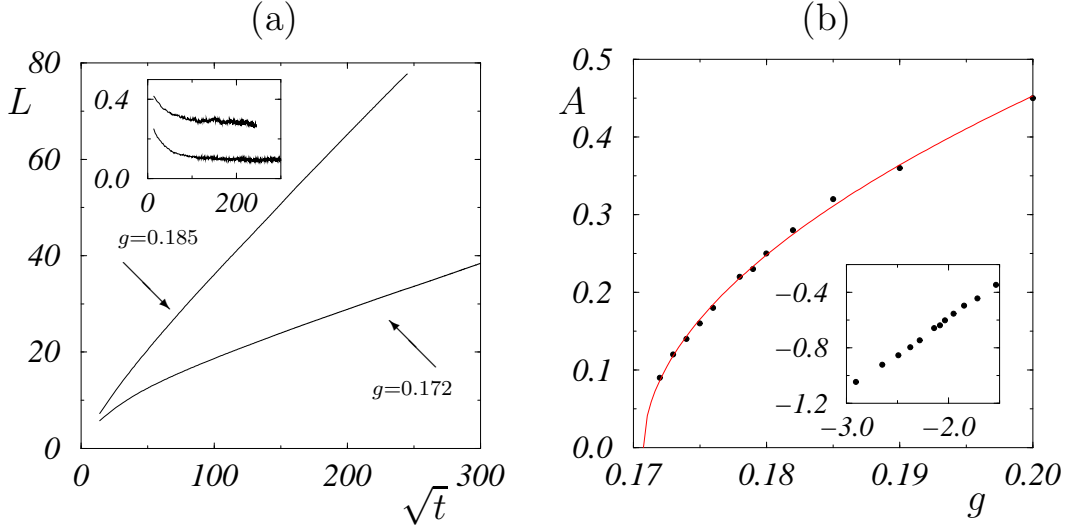


Fig. 5. Domain coarsening for our CML on a lattice of 8192^2 sites for two values of the coupling strength ($g = 0.172, 0.185$): (a) $L(t)$ vs. \sqrt{t} : normal coarsening is recovered after a transient of order 10^4 timesteps, so that, in the asymptotic regime: $L(t) \propto A(g)\sqrt{t}$. (b) Effective mobility $A(g)$ vs. g from simulations of lattices of linear size at least 4096^2 iterated during 10^5 timesteps. The fit $A(g) \simeq (g - g_e)^{1/2}$ allows one to estimate the threshold $g_e \simeq 0.171$ below which domain wall motion freezes. Insert: $\log A(g)$ vs $\log(g - g_e)$.

2.3 Large-scale domain growth

We thus went back to domain growth transients following random initial conditions, but with larger system sizes and longer simulation times than before. Fig. 5 shows the growth of $L(t)$ for a system of 8192^2 sites during 10^5 timesteps (runs performed on a parallel machine for a total of approximately 2000 cpu hours). A log-log plot of $L(t)$ reveals an increase of the local slope at long times, which, however, does not reach the “normal” value $1/z = 1/2$. On the other hand, the emergence of the normal growth behavior is clearly seen in a plot of L vs $t^{1/2}$, which becomes linear after approximately 10^4 iterations (Fig. 5a). The approach of the freezing transition at $g = g_e$ is therefore contained in the prefactor of the growth law $L(t) \propto A(g)t^{1/2}$. The effective “mobility” $A(g)$ seems to decrease continuously to zero. An excellent fit of the data is $A(g) \simeq (g - g_e)^w$ with $w = 0.52(3)$ (Fig. 5b), yielding the new estimate $g_e \simeq 0.171(1)$, but we cannot exclude a linear behavior near threshold which would yield a g_e value closer to our previous estimate in [4].

To sum up: domain coarsening in the CML studied here does show discrepancies with Eq. (3) (such as the ill-defined character of the mobility), but this seems to quantitatively influence the dynamics at early times only, so that normal growth behavior is recovered asymptotically at long times. It must be noted, though, that the effect of these features, specific to lattice deterministic systems, decreases rather slowly in time, as testified by the long crossover times recorded in our simulations.

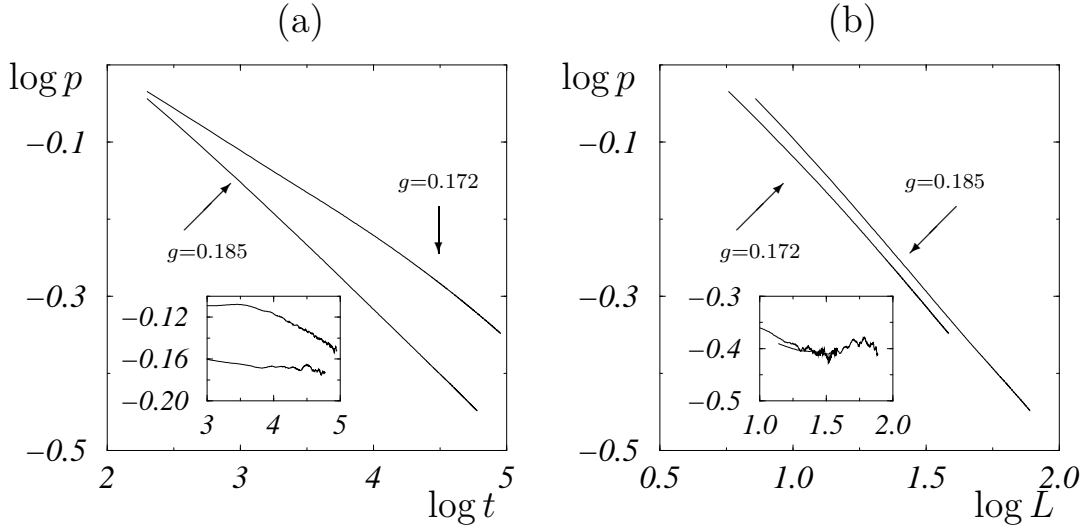


Fig. 6. From the same runs as those presented in Fig 5: (a): Decay of the persistent fraction $p(t)$ in log-log scales. (b): Same, but as a function of $L(t)$, the typical cluster scale. Inserts: local exponents.

3 Persistence issues

3.1 Persistence at late times

We now turn to the other set of results presented in [4], those related to the decay of $p(t)$, the fraction of persistent sites. Fig. 6a shows the decay of $p(t)$ for the same run as in Fig. 5. The behavior of the local exponent clearly changes at late times, in coincidence with the crossover observed in the growth of $L(t)$. The persistence exponents estimated in [4] thus reflect only the short-time behavior of the coarsening process.

Given that $L(t)$ reaches its asymptotic behavior rather late, one would ideally like to start measuring persistence from an initial time t_0 larger than the crossover time, so that the whole history of the system coming into account in the decay of $p(t)$ be situated in the asymptotic scaling regime. Unfortunately, this is basically out of reach of current computers' power, since it would require very large systems and their simulation over times much larger than t_0 .

On the other hand, it was noticed in [4] that the decay of the persistence probability recorded as a function of $L(t)$ is in a sense “more universal”, as it seems to be roughly independent of g . Those measurements were performed in the intermediate scaling regime where the growth of $L(t)$ is not normal. For later times and larger system sizes, the variation of p with L is seen to be better behaved than the simple decay of $p(t)$ (Fig. 6b). The local slope is approximately constant, and especially so after the crossover time for the growth of $L(t)$. We also checked that changing t_0 does not significantly influence the results. We can thus estimate a reliable persistence exponent, whose

variation with g is not significant given our numerical accuracy, and which takes approximately the value known for the TDGL equation, i.e. $\theta \simeq 0.20(2)$.

3.2 Generalized persistence

The behavior of persistence seems independent of g , or rather we cannot resolve the possible variation of exponent θ as we go from the threshold (near g_e) to the continuous-space limit. Moreover, although the estimated θ is close to the TDGL value, the error bar does not allow us to rule out the currently accepted value for the Ising model.

That θ is close to the TDGL value is not surprising if one considers that both our system and the TDGL equation are governed by deterministic dynamics. In the strong coupling limit of CMLs, in particular, interfaces evolve smoothly on a continuous space, and their motion is expected to resemble that of interfaces in TDGL. On the other hand, near g_e , strong lattice/anisotropy effects arise, yielding somewhat jittery interface motion, similarly to the discrete walls observed in Ising model.

Although persistence seems to depend only weakly on these lattice effects, it was recently proposed that these discrepancies become obvious when generalized persistence is considered [7]. In order to define generalized persistence, let us consider the time-average of the coarse-grained variable $\sigma_{\vec{r}}^t = \text{sign}(X_{\vec{r}}^t)$. At each point \vec{r} in space and starting from some initial time t_0 , it reads,

$$M_{\vec{r}}^t = \frac{1}{t - t_0} \sum_{t'=t_0}^t \sigma_{\vec{r}}^{t'} .$$

Generalized persistence $P(t; x)$ can then be defined as the probability that $M_{\vec{r}}^t$ has always remained above some threshold $x \in [-1, 1]$ [14]:

$$P(t; x) = \text{Prob} \left(M^{t'} \geq x; \forall t' \leq t \right) .$$

Generalized persistence is nothing but the “standard” persistence for the process $\text{sign}(M^t - x)$, and is thus also expected to decay algebraically in time with exponent $\theta(x)$. This provides a spectrum $\theta(x)$ of generalized persistence exponents, which contains, in particular, the “standard” persistence exponent $\theta(1) = \theta$.

As they discriminate in deeper details the possible paths followed by the variable $\sigma_{\vec{r}}^t$, generalized persistence exponents are expected to encode more information on the properties of interface motion than simple persistence and, in particular, to be sensitive to the influence of the jittery motion of interfaces [7].

Here we estimate $\theta(x)$ near and away from g_e . The measurement of generalized persistence has been carried out on the same runs as those presented previously. Spectra of generalized persistence exponents are displayed in Fig. 7. It should first be noted that these spectra are difficult to resolve numerically for small x values, since the corresponding data relies on the relatively rare spins which have spent most of their time in the phase opposite to their original phase. Thus, the data presented in Fig. 7 for this region can only have an indicative value (it is believed on general grounds, however, that $\theta(x)$ goes to zero as $(1+x)^\theta$ when $x \rightarrow -1$).

On the other hand, the normalized exponents $\theta(x)/\theta$ are rather well-resolved numerically near $x = 1$, and their behavior in this region is expected to characterize the short-scale motion of interfaces. The insert in Figure 7 shows $\theta(x)/\theta$ for our CML as well as for the Ising model and the TDGL equation. The normalized exponent spectra of our CML tend to the TDGL behavior as g goes away from g_e and approaches the continuous-space limit. One can notice, moreover, that our CML exhibits, under identical experimental conditions, the same value $\theta \simeq 0.204$ for both values of g presented here, despite the important change in the strength of discretisation effects. Thus, these effects are not felt at the level of simple persistence but are well captured, in a sense, by generalized persistence. Space discretisation involves at least two factors: the special behavior of interfaces (as seen in Section 2.2) and anisotropy. The results reported here do not allow one to separate them but imply that they seem to have only a marginal effect on θ . We note, further, that this is in agreement with recent studies of persistence in anisotropic partial differential equations in which no detectable change of θ with anisotropy could be recorded [15], but in disagreement with the claims of [16].

4 Conclusion

The results presented here have elucidated the surprising coarsening behavior of chaotic CMLs previously reported in [4], which we have shown to be representative only of the long, intermediate scaling region present in such deterministic systems. “Normal”, $z = 2$ phase-ordering is recovered at large time/length scales for all coupling strengths $g > g_e$.

In fact, both the growth law and the persistence exponent θ seem to be independent of g in this limit. Although our numerical data does not allow for a definitive statement about this, it provides an interesting case for the current debate considering which factors —space-discretization, anisotropy, chaos or probabilistic motion— are possibly influencing persistence exponents [16,7].

The general picture emerging for the phase-ordering properties of our CML

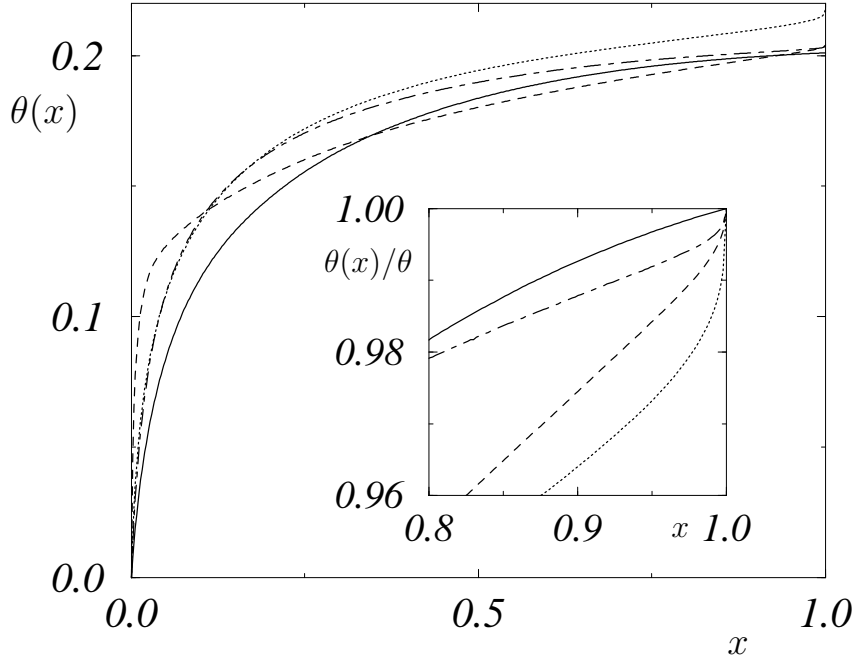


Fig. 7. Generalized persistence exponent spectra $\theta(x)$ for the TDGL equation ($t_0 = 100$, $L = 8192$, $dt = 0.2$, $dt/dx^2 = 1/8$, $t_{\max} \sim 16000$, solid line), our CML ($t_0 = 100$, $L = 8192$, $t_{\max} \sim 10^5$) at $g = 0.172$ (dashed) and $g = 0.185$ (dot-dashed), and the zero-temperature Ising model ($t_0 = 100$, $L = 16384$, $t_{\max} = 25000$, dotted). Insert: $\theta(x)/\theta$.

is as follows: while TDGL-like, smooth interface motion is recovered in the continuous-space limit, near g_e , important lattice/anisotropy effects yield interfacial properties somewhat similar to that of the Ising model; but these are only felt in generalized persistence scaling, confirming that this quantity captures details of interface dynamics. Generalized persistence spectra $\theta(x)$ show a significant qualitative change as g goes from g_e to the continuous-space limit, similar to the recently observed difference for $\theta(x)$ between the Ising and TDGL. Thus our CML, to some extent, interpolates between these two models.

In this regard, the ultimate question of the “true” asymptotic behavior of the phase-ordering properties of our CML must depend on further knowledge about the status of the commonly observed numerical differences in persistence properties between the TDGL equation and the Ising model. Future work should focus on resolving the question of observed discrepancy’s origin, possibly with the help of “intermediate” systems such as the CML studied here.

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